Specialist Mathematics

2015 Subject Outline

Stage 2

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Introduction

Purposes of the SACE

The South Australian Certificate of Education (SACE) is designed to enable students to:

* develop the capabilities to live, learn, work, and participate successfully in a changing world
* plan and engage in a range of challenging, achievable, and manageable learning experiences, taking into account their goals and abilities
* build their knowledge, skills, and understanding in a variety of contexts, for example, schools, workplaces, and training and community organisations
* gain credit for their learning achievements against performance standards.

Subject Description

Specialist Mathematics may be undertaken as a 20-credit subject at Stage 2 and is designed to be taken in conjunction with Stage 2 Mathematical Studies. Students who complete Stage 2 Specialist Mathematics with a C grade or better will meet the numeracy requirement of the SACE.

Mathematics is a diverse and growing field of human endeavour. Mathematics makes a unique contribution to the understanding and functioning of today’s complex society. By facilitating current and new technologies and institutional structures, mathematics plays a critical role.

Individuals require many aspects of mathematics in order to function adequately as members of society. The unprecedented changes that are taking place in the world will profoundly affect the future of today’s students. The effective use of technology and the processing of large amounts of quantitative data are becoming more important. Mathematics is increasingly relevant to the workplace and in everyday life. The study of mathematics provides students with the abilities and skills to thrive now and in the future.

Mathematics is much more than a collection of concepts and skills; it is a way of approaching new challenges by investigating, modelling, reasoning, visualising, and problem-solving, with the goal of communicating the relationships observed and problems solved.

Mathematics enables students to identify, describe, and investigate the patterns and challenges of everyday living. It helps students to analyse and understand the events that have occurred and to predict and prepare for events to come so they can more fully understand the world and be knowledgable participants in it.

Mathematics is a universal language that is communicated in all cultures. It is appreciated as much for its beauty as for its power. Mathematics can be seen in patterns in nature and art, in the proportions in architecture, in the form of poetry, and in the structure of music. Mathematics describes systematic, random, and chaotic behaviour; it is about relationships, exploration, intuition, and strategy.

Specialist Mathematics enables students to experience and understand mathematics as a growing body of knowledge for creative use in application to an external environment — a view of mathematics that students are likely to find relevant to their world. This subject deals with phenomena from the students’ common experiences, as well as from scientific, professional, and social contexts.

Students can gain from Specialist Mathematics the insight, understanding, knowledge, and skills to follow pathways that will lead them to become designers and makers of technology. The subject provides pathways into university courses in mathematical sciences, engineering, computer science, physical sciences, and surveying. Students envisaging careers in other related fields, including economics and commerce, may also benefit from studying this subject.

Capabilities

The aim of the SACE is to develop well-rounded, capable young people who can make the most of their potential. The capabilities include the knowledge and skills essential for people to act in effective and successful ways.

The five capabilities that have been identified are:

* communication
* citizenship
* personal development
* work
* learning.

The capabilities enable students to make connections in their learning within and across subjects in a wide range of contexts.

Aspects of all the capabilities are reflected in the learning requirements, the content, the assessment design criteria, and the performance standards. Specialist Mathematics empowers students to better understand and describe their world, and changes in it. As a result, students appreciate the role mathematics can play in effective decision-making. This subject also caters for students who want to continue to learn mathematics, and opens up a range of different career options by addressing aspects of the capabilities for work and learning. Although communication is an explicit feature of the assessment design criteria and the performance standards, the problems-based approach provides opportunities for students to develop aspects of the capabilities for citizenship and personal development.

Communication

In this subject students develop their capability for communication by, for example:

* communicating mathematical reasoning and ideas to a range of audiences, using appropriate language and representations, such as symbols, equations, tables, and graphs
* interpreting and using appropriate mathematical terminology, symbols, and conventions
* analysing information displayed in a variety of representations and translating information from one representation to another

justifying the validity of the results obtained through technology or other means, using everyday language, when appropriate

* building confidence in interpreting, applying, and communicating mathematical skills in commonly encountered situations to enable full, critical participation in a wide range of activities.

Students have opportunities to read about, represent, view, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing, and modifying ideas, attitudes, and beliefs about mathematics. Students are encouraged to use different forms of communication while learning mathematics.

Communication enables students to make connections between concrete, pictorial, symbolic, verbal, written, and mental representations of mathematical ideas.

Students develop the ability to explore, to make and test or prove conjectures, to reason logically, and to use a variety of mathematical methods to solve problems.

Citizenship

In this subject students develop their capability for citizenship by, for example:

* understanding how mathematics helps individuals to operate successfully in an emerging global, knowledge-based economy
* gaining knowledge and understanding of the ways in which mathematics can be used to support an argument or point of view
* acquiring mathematical skills that will enable students to become leaders in various fields of endeavour in society
* understanding the contribution of mathematics and mathematicians to society now and in the future
* learning to critique the ways in which the mass media present particular points of view
* understanding the mathematics involved in technologies and making informed decisions about their use.

In a time of major change, nations, states, and their citizens have to operate successfully in an emerging global, knowledge-based economy. Major social, cultural, and environmental changes are occurring at the same time as changing commercial relationships, and the introduction of new information and communication technologies and the more recently developed sciences and technologies. Mathematics plays an important part in all of these.

In Specialist Mathematics the main emphasis is on developing students’ knowledge, understanding, and skills so that they may use their mathematics with confidence as informed citizens capable of making sound decisions at work and in their personal environments.

Students are living in a rapidly changing world where decisions are based on quantitative understanding and reasoning. Therefore it is important that they value the necessity and the relevance of mathematics for lifelong learning.

Mathematics allows people to deal with aspects of reality and provides the language to describe certain phenomena. Students should be able to discuss mathematical ideas in a clear, concise manner.

Mathematics is contextual and relies upon agreements among people who use it. All citizens should learn to appreciate this aspect of mathematics as a worldwide intellectual and cultural achievement. Understanding the history of mathematics in their culture and using mathematics successfully celebrates this achievement and allows further evolution of mathematics.

Personal Development

In this subject students develop their capability for personal development by, for example:

* acquiring the capacity for inventive thought and problem-solving, using mathematical techniques
* gaining an appreciation of the value of mathematics to the lifelong learner
* making decisions informed by mathematical reasoning
* arriving at a sense of self as a capable and confident user of mathematics by expressing and presenting ideas in a variety of ways.

Students should be able to use mathematics as a tool to solve problems they encounter in their personal lives. Every student should acquire a repertoire of problem-solving strategies and develop the confidence needed to meet the challenges of a rapidly changing world.

Technology offers a wide and ever-changing variety of services to individuals and enterprises. It is important therefore that individuals have confidence in their mathematical abilities to understand the services offered and make informed judgments about them.

Work

In this subject students develop their capability for work by, for example:

* reaching an understanding of mathematics in a range of relevant work contexts
* understanding the role of mathematics in contemporary technological society
* gaining the mathematical knowledge and skills required for the particular pathway chosen by the student.

The mathematical skills required in the workplace are changing, with an increasing number of people involved in mathematics-related work. Such work involves increasingly sophisticated mathematical activities and ways of thinking. Although the use of information technology has changed the nature of the mathematical skills required, it has not reduced the need for mathematics.

It is important that students have the opportunity to gain an understanding of mathematics that will allow them to be designers of the future and leaders in various fields. They may be involved in product design, industrial design, production design, engineering design, or the design of new financial and commercial instruments.

The same considerations apply to the new sciences, and the new technologies they support. As systems for information-searching, data-handling, security, genetic design, molecular design, and smart systems in the home and at work become more sophisticated, users need to have a basic fluency in mathematics, and the designers of such technologies need to have an increasing understanding of mathematics.

Mathematics is a fundamental component of the success, effectiveness, and growth of business enterprises. Employees at various levels and in many types of employment are required to use their mathematical skills. Workers taking on greater responsibility for their own work areas use a wide range of mathematical skills. Some mathematical skills are used subconsciously because they are embedded in tasks.

Learning

In this subject students develop their capability for learning by, for example:

* acquiring problem-solving skills, thinking abstractly, making and testing conjectures, and explaining processes
* making discerning use of electronic technology
* applying knowledge and skills in a range of mathematical contexts
* interpreting results and drawing appropriate conclusions
* understanding how to make and test projections from mathematical models
* reflecting on the effectiveness of mathematical models, including the recognition of strengths and limitations
* using mathematics to solve practical problems and as a tool for learning beyond the mathematics classroom
* acquiring the skills to access and evaluate mathematical knowledge and applications.

The unprecedented changes that are taking place in today’s world are likely to have a profound effect on the future of students. To meet the demands of the world in which they live, students need to adapt to changing conditions and to learn independently. They require the ability to use technology effectively and the skills for processing large amounts of quantitative information. They need an understanding of important mathematical ideas; skills of reasoning, problem-solving, and communication; and, most importantly, the ability and the incentive to continue learning on their own.

Making connections to the experiences of learners is an important process in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students are able to value mathematics as useful, relevant, and integrated, and to confidently apply their knowledge and skills to making decisions.

Students need to solve problems requiring them to use prior learning in new ways and contexts. Problem-solving builds students’ depth of conceptual understanding.

Learning through problem-solving assists students when they encounter new situations and respond to questions of the type ‘How could I…?’ or ‘What would happen if…?’ Students develop their own problem-solving strategies by being open to listening, discussing, conjecturing, and trying different strategies.

Mathematical reasoning helps students to think logically and make sense of mathematics. Students are encouraged to develop confidence in their abilities to reason and justify their mathematical thinking.

Literacy in Specialist Mathematics

It is important that students are able to express, interpret, and communicate information and ideas. Specialist Mathematics gives students opportunities to grow in their ability to read, write, and talk about situations involving a range of mathematical ideas.

The ability to shift between verbal, graphical, numerical, and symbolic forms of representing a problem helps people to formulate, understand, and solve the problem, and communicate information. Students must have opportunities in mathematics to tackle problems requiring them to translate between different representations, within mathematics and between other areas.

Students learn to communicate findings in different ways, including orally and in writing, and to develop ways of illustrating the relationships they have observed or constructed.

Numeracy in Specialist Mathematics

Students who complete Stage 2 Specialist Mathematics with a C grade or better will meet the numeracy requirement of the SACE.

Being numerate is increasingly important in contemporary technological society. Students today require the ability to reason and communicate, to solve problems, and to understand and use mathematics. Developing these skills helps students to become numerate.

Students have opportunities to further develop their numeracy skills through the study of Stage 2 Specialist Mathematics. The problems-based approach, integral to the development of the mathematical models and the associated key ideas in each topic, ensures the ongoing development of mathematical knowledge, skills, concepts, and technologies in a range of contexts.

Becoming numerate involves developing the ability to understand, analyse, critically respond to, and use mathematical knowledge, skills, concepts, and technologies in a range of contexts that can be applied to:

* using measurement in the physical world
* gathering, representing, interpreting, and analysing data
* using spatial sense and geometric reasoning
* investigating chance processes
* using number, number patterns, and relationships between numbers
* working with graphical and algebraic representations, and other mathematical models.

Aboriginal and Torres Strait Islander Knowledge, Cultures, and Perspectives

In partnership with Aboriginal and Torres Strait Islander communities, and schools and school sectors, the SACE Board of South Australia supports the development of high-quality learning and assessment design that respects the diverse knowledge, cultures, and perspectives of Indigenous Australians.

The SACE Board encourages teachers to include Aboriginal and Torres Strait Islander knowledge and perspectives in the design, delivery, and assessment of teaching and learning programs by:

* providing opportunities in SACE subjects for students to learn about Aboriginal and Torres Strait Islander histories, cultures, and contemporary experiences
* recognising and respecting the significant contribution of Aboriginal and Torres Strait Islander peoples to Australian society
* drawing students’ attention to the value of Aboriginal and Torres Strait Islander knowledge and perspectives from the past and the present
* promoting the use of culturally appropriate protocols when engaging with and learning from Aboriginal and Torres Strait Islander peoples and communities.

Learning Scope and Requirements

Learning Requirements

The learning requirements summarise the knowledge, skills, and understanding that students are expected to develop and demonstrate through their learning.

In this subject, students are expected to:

1. understand fundamental mathematical concepts, demonstrate mathematical skills, and apply mathematical procedures in routine and non-routine contexts

2. practise mathematics by analysing data and any other relevant information elicited from the study of situations taken from social, scientific, economic, or historical contexts

3. think mathematically through inquiry, evaluation, and proof

4. make informed and critical use of electronic technology to provide numerical results and graphical representations, and to refine and extend mathematical knowledge

5. communicate mathematically and present mathematical information in a variety of ways

6. work both individually and cooperatively in planning, organising, and carrying out mathematical activities.

These learning requirements form the basis of the:

* learning scope
* evidence of learning that students provide
* assessment design criteria
* levels of achievement described in the performance standards.

Content

Stage 2 Specialist Mathematics is a 20-credit subject that consists of the following five topics:

* Topic 1: Trigonometric Preliminaries
* Topic 2: Polynomials and Complex Numbers
* Topic 3: Vectors and Geometry
* Topic 4: Calculus
* Topic 5: Differential Equations.

Each topic consists of a number of subtopics. These are presented in this subject outline, in two columns, as a series of key questions and key ideas side-by-side with considerations for developing teaching and learning strategies.

A problems-based approach is integral to the development of the mathematical models and associated key ideas in each topic. Through key questions teachers can develop the key concepts and processes that relate to the mathematical models required to address the problems posed. The considerations for developing teaching and learning strategies present suitable problems for consideration, and guidelines for sequencing the development of the ideas. They also give an indication of the depth of treatment and emphases required. This form of presentation is designed to help teachers to convey the concepts and processes to their students in relevant social contexts.

The key questions and key ideas cover the prescribed areas for teaching, learning, and assessment in this subject. It is important to note that the considerations for developing teaching and learning strategies are provided as a guide only. Although the material for the external examination will be based on the key questions and key ideas outlined in the five topics, the applications described in the considerations for developing teaching and learning strategies may provide useful contexts for examination questions.

Mathematics is a key enabling science for the technologies that are driving the new global economy. Much of the power of computers derives from their ability, in the hands of mathematically knowledgable people, to harness the subject in new and creative ways.

Specialist Mathematics presents three traditional topics, complex numbers, vectors and geometry, and the calculus of trigonometric functions, in a way that promotes their fundamental concepts as a paradigm for models of interacting quantities. The aim is to provide students with an appreciation of certain mathematical ideas that are both elegant and profound, and at the same time to allow them to understand how this kind of mathematics enables computers to model, for example, chemical, biological, economic, and climatic systems.

Specialist Mathematics presents ideas that are new to the student, and gives a new emphasis to familiar ones, by featuring the modelling capabilities of the topics presented.

First among these is the idea that functions describe time-varying quantities. The trace on a heart-monitoring machine and a seismograph record the history of the values of a time-varying quantity, look like the graph of a function, and can be interpreted in such a way. By this interpretation the derivative of the function describes the instantaneous rate of change of the values of the quantity. Students therefore acquire an interpretation of derivatives in addition to the usual ‘slope of the tangent’.

The trigonometric functions exemplify this interpretation. Their treatment in Stage 1 Mathematics is reviewed, as they are reintroduced as time-varying quantities associated with a point moving round a circle of unit radius at unit speed. Trigonometric functions are heavily emphasised because of their overwhelming application and importance as basic models of cyclical phenomena.

A second idea in Specialist Mathematics is that when several quantities need to be considered simultaneously they should be regarded as properties of a single entity of some kind. For example, pairs of quantities can be regarded as the Cartesian coordinates of a single point, or the components of its position vector, or the real and imaginary parts of a single complex number. A pair of time-varying quantities can therefore be regarded as a moving point, moving vector, or moving complex number, and each interpretation brings particular insights. The rates of change of a pair of quantities can equally well be regarded as the components of a vector or the real and imaginary parts of a complex number. The vector in this case describes the velocity of the moving point.

The third idea in Specialist Mathematics builds on the first two by treating cosine and sine as a single entity, rather than separately. When this approach is adapted to the calculus of cosine and sine, it is the pattern  satisfied by the pair of functions  that is important, treated as a relationship between the velocity  of a moving point and its position vector  It expresses the fact that the velocity of the circular motion defining cosine and sine is the rotation of the radius vector anticlockwise through a right angle. For complex numbers this rotation is implemented by multiplication by  and the identity  corresponds to the appearance of  in the pattern for the derivatives of cosine and sine. This indicates some of the interrelationships that Specialist Mathematics seeks to bring out by its particular approach to the three topics.

The immense modelling capability of mathematics derives from the ability to determine the complete time behaviour of a collection of quantities from laws by which they interact with or influence each other. When treated as functions, and their rates of change by derivatives, these laws are called ‘differential equations’.

The calculus of cosine and sine provides a fundamental example of this modelling capability. Moreover, their law of interaction  is not merely derivable from circular motion, it is characteristic of it. Any *two* quantities that interact by this law — whether two enzyme concentrations, or voltage and current, or velocity and displacement — can be represented by a point undergoing circular motion, and their time behaviour described using cosine and sine. The treatment of the calculus of trigonometric functions in Specialist Mathematics is designed to culminate in, and provide access to, this ubiquitous and useful model of basic cyclical behaviour and its simple variants. This is the purpose of the last topic, ‘Differential Equations’. This topic also highlights a review of the fundamental theorem of calculus from the point of view of differential equations, intended to deepen students’ perspective on indefinite integrals, and an explanation of the logistic functions that were introduced in Mathematical Studies, now derived from a simple law of evolution for self-interacting species.

The focus on new ideas and perspectives in Specialist Mathematics does not detract from the importance of the basic material. The arithmetic of complex numbers is developed and their geometric interpretation as an expansion of the number line into a number plane is emphasised. Their fundamental feature, that every polynomial equation has a solution over the complex numbers, is promoted and De Moivre’s theorem is exploited to find  roots.

Topic 3: Vectors and Geometry extends the two-dimensional ideas developed in Stage 1 Mathematics to the study of lines and planes in three dimensions, their intersections, and the angles formed by them. It also extends the study of circular geometry and tangents, and develops vector methods of proof. The calculus of trigonometric functions necessitates a review of the rules of calculus and their operation in this new context, providing the opportunity for increased skill and a deeper appreciation of how they work.

Specialist Mathematics puts students in a position to understand what lies behind computer models of two interacting quantities, represented as a point moving on a computer screen. Students are encouraged to explore the laws of circular motion with suitable computer packages, and to extend their experiments to more complex laws of interaction that pertain to competing biological species, simple enzymatic systems, or electrical feedback.

By these explorations, Specialist Mathematics opens up pathways to a variety of ‘mathematics futures’ including: geometry and vector methods beyond two and three dimensions to provide models of systems with many degrees of freedom; the general theory of differential equations, chaos theory, and dynamic systems to analyse the consequences of more complex laws of interaction. Students who are inspired to consider ideas such as these and become familiar with this kind of mathematics will be well placed to contribute to the new sciences and technologies, and to win a place for themselves in the new global economy.

Topic 1: Trigonometric Preliminaries

Subtopic 1.1: Graphs of Trigonometric Functions

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| What trigonometric knowledge is needed?Sine and cosine function values are the  and coordinates of a moving point on the unit circleAlgebraic identities satisfied by trigonometric functions and derived from the symmetry of the unit circle to establish special cases:*
*
*
*
*

and consequent identities such as*
*
*
*
*
*

Definition of odd function and even functionA brief treatment of the tangent function, including its period | Students review the definition of trigonometric functions as developed in Stage 1 Mathematics Subtopics 14.4 and 14.5 (sine function), 14.6 and 14.7 (cosine function), and 14.8 (tangent function), and the concepts of radian measure and arc length. The  and  coordinates of the moving point are considered as functions of the arc length  (Note that this is a radian measure of the angle subtended by the arc at the centre of the circle.)Students can derive these symmetry properties from symmetry operations on the model of a point moving round the unit circle, and interpret them as properties of the graphs of cosine and sine.It can be seen from the identities that cosine is an even function and sine is an odd function. This is compared with instances of these properties for polynomial functions and possibly the absolute value function. |

Subtopic 1.2: Properties of Trigonometric Graphs

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How can transformations be represented trigonometrically?Horizontal and vertical dilations, translations, and/or reflections, and their effect on trigonometric functions and their graphs, including a brief treatment of all three reciprocal functions:  | Exploration, through the use of graphing technology and interactive geometry, of the effect of:* vertical dilations on the equation and amplitude of trigonometric functions
* horizontal dilations on the equation and period of trigonometric functions
* vertical translations on the equation and range of trigonometric functions
* horizontal translations on the equation and range of trigonometric functions
* reflection in the  on the equation and graph of trigonometric functions.

The graphs and periodicity of the reciprocal trigonometric functions can also be considered using electronic technology. |

Subtopic 1.3: Trigonometric Identities

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| What trigonometric identities for sine and cosine functions are used?* Addition formulae
* Double angle formulae
 | Students derive the formula for  by the dot product (see Subtopic 13.9 of Stage 1 Mathematics). Students compare this derivation with the approach using the cosine rule (see Subtopic 14.7 of Stage 1 Mathematics). The other addition formulae are derived using the properties established in Subtopic 1.1.The identitycould be considered at this point for later use in Topic 4: Calculus. |

Topic 2: Polynomials and Complex Numbers

Subtopic 2.1: Complex Numbers

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Why were complex numbers ‘invented’? | Revision of the introduction of complex numbers from Subtopic 10.1 of Stage 1 Mathematics, in order to be able to describe solutions to all quadratic equations with real coefficients. (Students note that the sum and product of the roots of a real polynomial are real.)Analogy can be drawn with the extension of the natural numbers to the integers, integers to rationals, rationals to reals — in each case with a view to ensuring that certain kinds of equations have solutions.A parallel is drawn with the arithmetic of surds and how surds arise — for example, solving the quadratic equation leads to numbers of the form  |

Subtopic 2.2: Complex Conjugation

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Calculating with complex numbers and their conjugates | Revision of addition, multiplication, and division of complex numbers from Subtopic 10.1 of Stage 1 Mathematics. Since the emphasis in this topic as a whole is intended to be on the geometric representation of complex numbers, this revision of basic operations is as brief as possible. Teachers may wish to introduce the Cartesian plane representation as detailed in Subtopic 2.4 at the same time as the revision.  |

Subtopic 2.3: Inductive Argument

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How are the conjugates of sum and product of any number of complex numbers found? | Students must interpret, with emphasis on inductive argument, statements such asand as infinite lists of statements labelled by  From this perspective they should consider strategies for deriving any statement in the list from the previous one. |
| Examples, in this context, using induction as a method of argument to establish generalisations | Induction is used to establish generalisations of the triangle inequality, that the modulus of a product is the product of the moduli, and to prove De Moivre’s theorem (this subtopic and Subtopics 2.4, 2.5, 2.6, and 2.7).There are some examples of this type of argument later in this subject outline. There are also examples in connection with the derivative of a sum of  functions being the sum of the derivatives, and the general formula for the derivative of (see Subtopic 2.6 of Stage 2 Mathematical Studies).Note the special case of the product formula which arises when all the  are equal.The following example provides two possible approaches to an inductive argument:Let  Use an inductive argument to prove that, if the  derivative of  is denoted by  thenSolution:First check the truth of the statement for the first value of  as required. |
|  | For the next step,either* assume  and then differentiate again to get

as required,ordifferentiate  to getthen again, and so on, with the power of  and the size of the factorial increasing, and the power of  decreasing, at each step. The pattern is described by the formula  as required. |

Subtopic 2.4: The Complex Number Plane

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How can complex numbers be represented geometrically?* Cartesian form
* Conjugate
* Modulus
 | The Cartesian plane as extension of the real number line to two dimensions.Correspondence between the complex number  the coordinates  and the vector  It is noted that complex number addition corresponds to vector addition via the parallelogram rule (Subtopic 13.7 of Stage 1 Mathematics). To this extent complex numbers can be regarded as two-dimensional vectors with rules for multiplication and division to supplement addition and subtraction. |
| Geometric notion of  as the distance between points in the plane representing complex numbers | Relative positions of  and its conjugate; their sum is real and difference purely imaginary. |
| Triangle inequality | The triangle inequality is a content thread that also appears in other parts of this subject outline, and the related ideas should be noted. Extension to the sum of several complex numbers can be argued by induction as noted above. |

Subtopic 2.5: Polar Form

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Is the Cartesian form always the most convenient representation for:* describing sets of points in the plane?
* multiplying and dividing complex numbers?
 |  |
| The properties  and  They are the basis on which multiplication by is interpreted as dilation by  and rotation by  | These properties make polar form the most powerful representation for dealing with multiplication.Students observe that the real and imaginary parts of the identity  are the addition of angles formulae for cosine and sine. Thus complex multiplication encodes these trigonometric identities in a remarkable and simple way. The geometric significance of multiplication and division as dilation and rotation should be emphasised, as should the geometric interpretation of modulus as distance from the origin. Geometry from Stage 1 Mathematics can be used — for example,using properties of a rhombus to determine the polar form of  from that of  |
| Multiplication by as anticlockwise rotation through a right angle | Although this follows from the addition formula for  it should also be demonstrated directly:  shows that multiplication by sends  which can be shown to be the effect on the coordinates of rotating a point about the origin anticlockwise through a right angle. |
| Conversion between Cartesian form and polar form | The conversions  where along with  and  and their use in converting between Cartesian form and polar form.Calculators can be used, both to check calculations and to enable students to consider examples that are not feasible by hand. |
| Graphical interpretation and solution of equations describing circles, lines, rays, and simple spirals, and inequalities describing associated regions; obtaining equivalent Cartesian equations and inequalities where appropriate | Graphical solution of equations and inequalities such as strengthening geometric interpretation. An important aim is the conversion of such equations and inequalities into Cartesian form in the case of circles and lines (as a link with work in other subtopics), through geometric understanding of the descriptions used above directly in terms of modulus and argument. Dynamic geometry software can be used as an aid. Students could investigate various polar graphs, using graphing technology. |
| The utility of the polar form in calculating powers of complex numbers  | The properties  and  should be argued by induction, as noted above.  |
| * De Moivre’s theorem for positive integral *n* (argument by induction)
 | Note also the special case  when all the  are equal, and  when all the are equal, which is De Moivre’s theorem for positive integral  |
| * Extension to negative integral powers and fractional powers

Solution of  ( real or purely imaginary), in particular the case  | Extension to negative powers viaand to fractional powers viaAs a particular example of the use of De Moivre’s theorem, the symmetric disposition of the  roots of unity in the complex plane should be appreciated. The fact that their sum is zero can be linked in the vector section of the subject outline to the construction of a regular  |

Subtopic 2.6: An Application — Quadratic Iteration

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How do simple quadratic iterations on complex numbers affect position in the plane? | This key question involves considering the geometric and algebraic properties of complex numbers, and examining the effect of repeated operations (iteration) on the position of the final result in the Cartesian plane. Using interactive geometry, spreadsheets, and/or a graphics calculator are possible strategies for supporting concept development here. |
| Position relative to the unit circle* Iterating the transformation  and the effect on the final result for values of  with:

 | Most students at this level would appreciate the effect of repeated squarings on a real number. The key ideas involved allow exploration of such concepts on the field of complex numbers. Students could explore the effect of repeated squarings algebraically, using Cartesian form for the first few results and De Moivre’s theorem for the higher powers. The use of electronic technology would visually and numerically support these explorations, allowing students to see a sequence of results for a range of complex numbers. The effect of the process on the modulus and argument of the results and the difference in behaviour for points on, in, and outside the unit circle are pertinent observations here. |
| Iterating the transformation* Introduction of the complex constant *c* into the transformation above has a marked effect on the iteration process. Behaviours to note are:

invariant points (one-cycles)cyclesconvergencedivergencechaotic behaviour.The criterion for divergence (escape criterion) of the iteration  with  for some  | Pen-and-paper manipulation for simple examples of invariant points, as well as two-cycles and three-cycles, could be followed by further investigations into these and other properties with the aid of electronic technology.Use of one of the commonly available Mandelbrot software programs would be a good starting point for this whole subtopic. After the Mandelbrot set has been explored visually, a more analytical exploration could be made, using electronic technology to plot orbit diagrams. In this way the Mandelbrot set could be explored, with students noting the different behaviour inside, near, and outside the chaotic boundary of the set. The relevance of the circle  and the role of the lobes in the Mandelbrot set could also be explored in this way.Students who want to investigate further examples of complex iteration could look at Julia sets and/or the fractal produced by iterating the so-called ‘Newton’s method transformation’, |

Subtopic 2.7: Fundamental Theorem of Algebra

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Operations on real polynomials: polynomials can be added and multipliedWhat can be said about division of polynomials? (It is sufficient to consider polynomials of degree )  | Review of the multiplication process. The division algorithm for polynomials. Polynomial long division technique. The use of undetermined coefficients and equating coefficients in factoring when one factor is given.  |
| * Roots, zeros, factors
* Remainder theorem, with proof; its use in verifying zeros
 | The correspondence between the roots of a polynomial equation, the zero of a polynomial, and the linear factor of a polynomial should be understood. |
| * Factorisation of cubics and quartics with real coefficients (given a zero), using long division or the multiplication process/inspection

With the introduction of complex numbers, all real polynomials have a factorisation into linear factors; this is the statement of the fundamental theorem of algebra | The existence of a formula for the roots of a cubic and a quartic, and some of the associated history, could be mentioned as background.Real polynomials can be factored into real linear and quadratic factors, and into linear factors with the use of complex numbers.The connection between the zeros and the shape and position of the graph of a polynomial should be explored. There should be many opportunities to make use of graphing technology.Special examples:  as solved above by De Moivre’s theorem; the factorisation of  and so on, as an illustration of the use of the remainder theorem. The statement of the fundamental theorem can be considered to answer the question ‘Why were complex numbers “invented”?’ Though not every real polynomial of degree has real zeros, in the field of complex numbers every real or complex polynomial of degree has exactly zeros (counting multiplicity). |

Topic 3: Vectors and Geometry

Subtopic 3.1: Algebraic and Geometric Treatments of Three-dimensional Vectors

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| What is a vector? | Revision of vectors as directed line segments (arrows) in space (generalising from the two-dimensional treatment in Stage 1 Mathematics). |
| What is a parallelogram in three dimensions?Equality of vectorsCoordinate systems and position vectors; components | Recall that vectors are equal if they form opposite sides of a parallelogram. The notion of a parallelogram in three dimensions requires a discussion of when the opposite sides of a quadrilateral are coplanar so that it is not twisted; the simplest criterion is that its diagonals intersect. Applications (e.g. navigation and force) as encountered in Stage 1 Mathematics can readily be extended to three dimensions.  |
| Operations with vectorsAddition of vectors, multiplication by scalars, length of vectors: both geometrically and algebraically | Contexts and applications as mentioned above. |
| Parallel vectors, collinearity, ratio of division of a line segment (internal and external)Can vectors be multiplied together?What is the meaning of the product?Scalar (dot) product and vector (cross) product: their properties and their interpretation in context | Ratio of division is needed in the treatment of Bézier curves (see Topic 4: Calculus). These two operations on pairs of vectors provide important geometric information. The scalar and vector products should be treated in terms of coordinates and of length and angle. Conditions for perpendicularity and parallelism, and construction of perpendiculars. |
| Cross-product calculation using the determinant | Though determinants may not have been encountered previously, this is a suitable place for their introduction, since they arise naturally in the formulae for area and volume. (Note that the area of a parallelogram in two dimensions can be expressed as a  determinant in terms of the coordinates of the defining vectors.) (Determinants are treated in Subtopic 3.4 of Stage 2 Mathematical Studies.) |
|  | The cross-product of two vectors  and  in three dimensions is a vector, mutually perpendicular to  and to  whose length is the area of the parallelogram determined by  and  The right-hand rule determines its sense. The components and the vector itself may be expressed using determinants. (Applications of scalar and vector products include work and moments.) |
| When and how is the scalar triple product used?* Volume of a parallelepiped
* Test for coplanarity
 | Scalar triple product as volume of a parallelepiped, again using determinant notation; test for coplanarity. |

Subtopic 3.2: Lines and Planes

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How is the equation of a line in three dimensions written?* Vector, parametric, and Cartesian forms
 | Geometric considerations lead to the vector equation of a line; from this can be derived the parametric form and (less importantly) the Cartesian form. Exercises should highlight the construction of parallel lines, perpendicular lines, and the phenomenon of skewness.Computation of the point of a given line that is closest to a given point; distance between skew lines. |
| The next simplest subset of three dimensions is a plane. How can a plane be described by an equation?* Cartesian form
 | The equation of a plane, also developed using geometric ideas as a vector equation from which the Cartesian form is derived. |
| Relationships between lines and planes | Intersection of a line and a plane, lines parallel to or coincident with planes. Computation of the point of a given plane that is closest to a given point. |
| Intersections of planes: algebraic and geometric descriptions | Finding the intersection of a set of two or more planes amounts to solving a system of linear equations in three unknowns; this is covered in Stage 2 Mathematical Studies. The geometric interpretation in the case of unique solution, infinite solution, or no solution should be discussed (but may need to be delayed until the end of the program to fit in with the timing of Stage 2 Mathematical Studies Topic 3: Working with Linear Equations and Matrices). |
| Angles formed by lines and planes | Calculation of the angle between lines, even if they are skew, and of the angle between a line and a plane, and the angle between two planes. |

Subtopic 3.3: Geometry of Circles and Tangents

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Circles and their tangents arise often in applications involving motionWhat basic properties should be familiar?* The radius is perpendicular to the tangent at the point of contact
* The perpendicular from the centre of a circle to a chord bisects the chord
* Tangents from an external point are equal in length
* The angle in a semicircle is a right angle
* The angle at the centre is twice the angle at the circumference
* Angles in the same segment are equal
* Alternate segment theorem

Cyclic quadrilaterals and concyclic points | Revision of theorems on angles in circles and tangents.Formal proofs of these theorems are not required, although the informal process of logical justification should be rigorous, with the emphasis on clearly communicating the sound justification of a general rule.(Note: Theorems on intersecting chords are not required.) |
| Simple deductive problems involving circles, tangents, and cyclic quadrilaterals | Examples will normally be expected to involve only one circle. |

Subtopic 3.4: Vector Methods of Proof

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| The use of vector methods of proof, particularly in establishing parallelism, perpendicularity, and properties of intersections | Students should see some examples of proof by vector methods, and appreciate their power.Suitable examples include:* the angle in a semicircle is a right angle
* medians of a triangle intersect at the centroid.

The result:If then if  are not parallel. |

Topic 4: Calculus

Subtopic 4.1: Functions and Quantities Varying with Time

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How exactly do functions describe time-varying quantities? | The trace on a heart-monitoring machine and a seismograph record the history of the values of a time-varying quantity, and look like the graph of a function. Students can explore how a seismograph or the trace on a heart-monitoring machine is produced in a way that connects directly with the graph of a function. For example, consider a vertical line moving from left to right at unit speed on a Cartesian plane, and carrying a point whose directed height at any instant gives the value of the quantity of interest. Such a model may well have been introduced to produce the graphs of cosine and sine from the circular motion model (see Subtopic 1.1, or Subtopics 14.4 to 14.7 of Stage 1 Mathematics). Their interpretation as functions of time and the production of their graphs in this fashion are well worth reviewing at this point.  |
| The derivative of a function as the instantaneous rate of change of the time-varying quantity that the function describes | Given a chord on the graph of the function, its ‘rise’ gives the change in the value of the quantity, and its ‘run’ gives the interval of time over which the change took place. The gradient of the chord is therefore the average rate of change over this interval of time. The gradient of a tangent, as a limiting value of the gradient of chords, naturally measures the rate of change of the quantity at the instant determined by the point of contact (connecting with Subtopic 2.6 of Stage 2 Mathematical Studies). |

Subtopic 4.2: Pairs of Uniformly Varying Quantities and Their Representation as a Moving Point

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| How can a pair of time-varying quantities be modelled as a point moving in a plane? | Given two time-varying quantities, their values at any instant can be interpreted as the Cartesian coordinates  of a point moving in the plane. |
| When the quantities vary uniformly with time, what kind of curve do they trace out?* Coordinate representation (parametric)
* Vector interpretation
 | Being uniform, the quantities are functions of the form  These describe the parametric equation of a line in two dimensions. This is understood with reference to the vector and parametric equations of lines in three dimensions studied in Subtopic 3.2. The vector form of the two-dimensional equation isEmphasis is placed on the interpretation of the parameter as time. It is noted that vector  gives the change in position vector occurring in each unit of time, and that this is the fundamental notion of velocity.Interpretation of complex numbers (optional).The pair  can also be considered the real and imaginary parts of a moving complex number  Then  where  and  Note that  is increased by the complex number  in each unit of time. |
| Examples involving pairs of uniformly varying quantities | Students may like to consider examples such as production costs with two components, capital and labour, and a simplistic model of predator–prey relationships, defined by two uniformly varying populations. |

Subtopic 4.3: Pairs of Non-uniformly Varying Quantities — Polynomials of
Degree 2 and 3

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| What curves are traced out by a moving point  in which the functions  and  are polynomials of degree 2 and 3?Examples of applications to:* objects in free flight
* Bézier curves

and equivalent examples should be considered. | The position of an object in free flight is given by the equations where  is the initial position,  is the initial velocity, and  is the acceleration due to gravity. Students observe the parabolic shape of the curve. This equation can be used to answer questions such as: ‘At what angle should a basketball be thrown to score a goal?’ or ‘At what angle should a cricket ball be struck to clear the fence?’Degree 3 polynomials feature in computer-aided design. The use of Bézier curves attempts to mimic freehand drawing using cubic polynomials, and motion along Bézier curves is used to simulate motion in computer animations. They are constructed using four control points, two marking the beginning and end of the curve and two others controlling the shape. They can be constructed with interactive geometry software:(www.moshplant.com/direct-or/bezier/)Although the curves can be drawn interactively and intuitively by the designer, an accurate mathematical description is needed for: editing the curve; zooming in by the program; passing the design on to be printed or processed by specialised machining equipment. |

Subtopic 4.4: Related Rates, Velocity, and Tangents

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Where two functions of time  are related by  their rates of change are related by the chain rule | Examples of calculating  and the derivative of  For instance, might be the length of the side of a square that is changing over time, and  its area. Or they might be the length of the side of a cube and its volume. |
| The notation for derivatives concerning time is: For a moving point  the vector of derivatives  is naturally interpreted as its instantaneous velocity | For uniform motion the velocity  has for its components the rates of change of the components of the position vector. The idea that  is interpreted as the instantaneous velocity vector of the moving point  is a natural extension of this idea, in which average rates of change are replaced by instantaneous rates of change, or derivatives. If, given a function  and a function of time  you set  then the moving point  travels along the graph of  |
| The velocity vector is always tangent to the curve traced out by a moving point | The chain rule  shows that the ratio of  is  and that the velocity vector is tangent to the graph. Since effectively every curve is the graph of a function (by rotation if necessary), the velocity vector is always tangent to the curve traced out by a moving point. |
| Parametric equations of tangents to parametric curves | At time  the moving point passes through  and the velocity vector  is tangent at this point to the curve traced out. The parametric line  passes through  at time  and has velocity  so it is tangent to the curve at this point. Students can use this formula to calculate the parametric equations of tangents to parametric curves, for example, Bézier curves. |
| Speed of the moving point as magnitude of velocity vector, that is | Calculation of the speed of projectiles or of points moving along Bézier curves. |

Subtopic 4.5: Derivatives of the Circular Functions

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Cosine and sine considered as functions of time | Students review from Subtopic 1.1 their understanding of cosine and sine in terms of the circular motion model, this time with an emphasis on the coordinates of the moving point as functions of time. |
| The derivatives of cosine and sine explained in terms of the relationship  derived from circular motion | By inspecting the graphs of cosine and sine, and the gradients of their tangents, students can conjecture that Emphasis is placed on the pattern  displayed by these formulae and its explanation in terms that the velocity vector  is obtained from the position vector  by an anticlockwise rotation through a right angle. This follows because both velocity vector and position vector have length  the motion is anticlockwise, and the velocity vector is tangent to the unit circle and therefore perpendicular to the position vector  (see Subtopic 3.3).Rotating the vector  anticlockwise through a right angle produces  The geometric explanation of this fact will have been given when interpreting multiplication by the complex number as anticlockwise rotation through a right angle (see Subtopic 2.5). |
| Motion round larger and smaller circles | Looking ahead to Subtopic 5.3, students are encouraged at this point to consider the motions  which also satisfy the pattern  They describe a point moving round another circle shadowing the point on the unit circle. |
| Angular velocity, and moving with different speed round the unit circle | The speed of the point is equal to the radius per unit time (i.e. 1 radian). Students might also interpret  as shadowing a different point on the unit circle that is always  units in advance of the moving point. Students argue geometrically why changing the speed of the point on the unit circle to  changes the motion to  On the other hand its velocity should be magnified by a factor  so that  Students are introduced to the use of the chain rule with cosine and sine by verifying the consistency of these two observations. They explore and interpret the variants  |
| The derivative of  from first principles | The calculation of the derivatives of cosine and sine through circular motion presumes that these derivatives exist. A first-principles calculation from the limit definition provides a proof that the derivatives exist, and allows students to see the results in the context of the ‘gradient of tangent’ interpretation of derivatives from which the limit definition is derived. This traditional calculation proceeds by first computing the derivative of sine at zero, which amounts to the limitwhich is derived graphically, numerically, and geometrically. The derivatives of the sine and cosine functions in general can be derived from this special case using addition of angles formulae (see Subtopic 1.3). |

Subtopic 4.6: Calculus of Trigonometric Functions

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Computing the derivatives of trigonometric functions and functions involving them | Using the rules for differentiation, the derivatives of all the trigonometric functions can be determined as well as composite functions involving the trigonometric functions. Higher order derivatives involving trigonometric functions such as the  derivative of  can be considered to give further examples of argument by mathematical induction. |
| Applications of the derivative involving:* simple rates of change
* maximum and minimum problems
* further related rates.
 | The determination of rates of change of quantities such as the:* height of a piston
* voltage at an electrical outlet.
 |
| Indefinite integrals and definite integrals of functions of the type Extension of the substitution method (covered in Stage 2 Mathematical Studies) to simple examples involving trigonometric functionsApplications to the calculations of areas, distance travelled, and so on | The maximum and minimum values in simple problems involving trigonometric functions (e.g. the maximum length of a ladder that can be carried round a corner or in a corridor, or the maximum or minimum height of a tide).The calculation of areas under the graph of trigonometric functions:* using the fundamental theorem of calculus
* with  as an example of where numerical methods are required to determine the area.
 |

Topic 5: Differential Equations

Subtopic 5.1: Introduction to Differential Equations

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| What is a differential equation?A differential equation is an equation that connects the rate of change of an unknown quantity with the values of known quantities, particularly its own | Students review the law of exponential growth from Stage 2 Mathematical Studies, interpreting it as stating that the rate of change of a quantity is some fixed multiple of (i.e. proportional to) its own value. Describing the quantity as the function  and the constant of proportion as  this states that  Possibilities for  can be described in terms of the exponential function. Students explore the question of what all the possibilities are. |
| In terms of functions, it is an equation between the derivative of the function, the independent variable, and the function itself | Examples such as population growth, as well as Newton’s law of cooling, radioactive decay, and simple capacitor/resistor circuits, are important for emphasising the mathematical modelling role of differential equations and the ability to determine from them the time behaviour of quantities. |
| Use of the notation  to denote the derivative of the function  | Note that  is often used to denote the derivative of  that is,  regardless of the label, name, or interpretation of its independent variable as time or whatever else.  |
| Review of integration as an example of solving simple differential equations | Students will gain a much deeper understanding of the fundamental theorem from a review of it in terms of differential equations. The quantity in this case is the area  of the strip under the graph of a function  swept out by a line moving from left to right at unit speed, starting from some point  The fundamental theorem of calculus states that the rate of change of this quantity at any instant is equal to the height of the leading edge of the strip (i.e. to the function value at that instant). This example may well have been used in Subtopic 2.13 of Stage 2 Mathematical Studies, to motivate study of the fundamental theorem. Integration is the process of finding solutions to the differential equation  these solutions are called ‘indefinite integrals’.  |
| Initial conditions and their use in specifying solutions | Students are aware that differential equations rarely have one solution, and usually have many; the fundamental theorem is an appropriate place to emphasise the fact. However, because the strip begins at  it is known that  This is called an ‘initial condition’. Solutions are often determined by initial conditions. In this case, if  satisfies  then the difference between  and  has derivative zero and must be constant.Comparing their values at  that difference must be  This produces the formula for areain terms of an indefinite integral  |
| How can the information about the derivative of a function be described? | An equation  indicates the slope of the graph at each point  but not the value of  A line of gradient  can be drawn at each point on each vertical line and one of these is the tangent line to the graph. This family of lines, one through each point in the plane, is called the ‘slope field’.  |
| Reconstruct a graph from a slope field both manually and using graphics software | Given an initial value, say,  for  at the slope field indicates the direction in which to draw the graph. These ideas should be displayed using graphics calculators or software, and students invited to consider how the computer or graphics calculator might have traced the curves — for example, by following each slope line for a small distance, then following the slope line at the new point. These graphical results are compared with known solutions from integration, such as the solutions  for the equation  What role does  play in the geometric picture? What is the key feature of a family of curves of the form  |
| Construct tables of approximate values for the solution numerically, using Euler’s method | Note that Euler’s method is most useful in the many cases when exact solutions cannot be found by integration. |

Subtopic 5.2: Separable Differential Equations

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| Separable differential equations* Differential equations expressible in the form  examples  and
 | The exponential equation  is the simplest example of a separable differential equation. Other examples such as  arise from Newton’s law of cooling or as models of the spread of rumours. They are solved in terms of exponential functions. The family of solutions, and the use of initial conditions to determine which one describes a problem, is emphasised.  |
| The logistic differential equation  | The logistic function features in Subtopic 2.12 of Stage 2 Mathematical Studies, where it is inferred from data. The differential equation provides an explanation as to why it arises in terms of a law of evolution.  represents a total population (of molecules, of organisms, etc.) of which are active or infected and  are not. To create more of the ‘actives’, an active and an inactive must meet. The rate of meeting is proportional to the product of the actives and inactives, and this will determine the rate of increase  of the actives. |
| Method of solution of separable differential equations | If  is an indefinite integral of  and  of  then differentiating  (using the chain rule on the left-hand side of the equation) reproduces the differential equation. Solving the equation  for  gives a function that therefore satisfies the equation above, and hence the differential equation. To carry out this method for the logistic differential equation, students will need to check the identity |

Subtopic 5.3: Interacting Quantities, Systems of Differential Equations, and Cyclical Behaviour

| Key Questions and Key Ideas | Considerations for Developing Teaching and Learning Strategies |
| --- | --- |
| A system of differential equations: specifying an interaction between two time-varying quantities means specifying the rate of change of each in terms of their two values. Describing the quantities by functions, the derivative of each one is given by a formula in the two functionsExamples should be restricted to the case of two equations for two functions | This key idea is explained and reinforced by a variety of simple examples from a variety of areas of knowledge such as biology, chemistry, electronics, mechanics, and climate. This is one of the most important points of contact between mathematics and other areas of knowledge, and a critical place where the power of its modelling capabilities is demonstrated by a universal model that applies across a wide range of knowledge. In order to exhibit this, both students and teachers must acquaint themselves with some of the key ideas and elementary quantitative relationships of these other areas of knowledge.  |
| It is not intended that students have knowledge of the method of solving second order differential equations | For example, voltage and current are familiar concepts. When a time-varying current flows through a coil it induces a voltage across the coil, against the flow and in proportion to its rate of change. This quantitative relationship can be expressed as  If the current is produced by discharge from a capacitor then, via another simple quantitative relationship,  where  is another constant. These two relationships constitute a law of interaction between, or a system of differential equations for, the voltage and current in a capacitor–coil circuit, the basic device for producing radio signals.When a spring is stretched or compressed, it experiences a force  in proportion to its extension  and in the opposite direction. Its velocity  is the rate of change of this extension so that  If a weight of mass  is attached, then its momentum is  and, by Newton’s second law,  The definition of velocity thus combines with Newton’s second law to provide a law of interaction between, or a system of differential equations for, the extension and velocity of the spring, the simplest of oscillatory devices whose variants are the basis of the mechanical watch.Examples from a variety of areas of knowledge, whether these or others chosen to appeal to the interests of students and teachers, are introduced and discussed. |
| The differential equations in the simplest examples have the formIn the special case where  it is known that  satisfy What other functions satisfy this case?The functions provide a solution that starts atSuitable choice of  and  to design a solution with prescribed initial values of  and  | It is easy to check that the given functions satisfy the system of differential equations. Students explain how they might have been led to predict this.The system  describes a moving point whose velocity vector is an anticlockwise rotation of its position vector through a right angle. Since its velocity is perpendicular to its position vector, its motion would be expected to be circular. Dilating  by a factor  gives  which describes a circular motion on a circle of radius  with speed  and starting on the *x*-axis. describes the same circular motion but with its clock delayed by  units of time, so that it starts at  In order to start at some prescribed point  is set so that  (this is conversion from Cartesian into polar form, as in Subtopic 2.5).  |
| What system of differential equations results from motion round a unit circle at some speed  not necessarily equal to 1? What functions might satisfy the system?On a circle of radius  a point moving according to  moves at a speed of or  radii per unit of time, or  radians per unit of timeConversion between ‘angular velocity’  frequency, and the period of the circular motion | Another variant on the circular motion that defines cosine and sine is a point travelling round a unit circle at some speed  not necessarily equal to 1. The velocity vector  of such a motion is perpendicular to the position vector, and must be some multiple of  This vector has unit length on the unit circle, but the velocity, which is  there, must be  and the system of differential equations must be  Speeding up by a factor of  can be achieved by replacing  with  in functions of time. This suggests that  satisfies the system. This is easily confirmed, and such a solution can be designed to have any prescribed starting point. |
| How might solutions of systems  be constructed when  are not equal? | To satisfy this system, students design solutions of the form  and with a prescribed starting point. They use graphics software to explore the motion of the point, and recognise that the paths traced out are ellipses. |
| The ‘interaction constants’  determine the angular velocity  and the ratio of  to  which measures the distortion of the ellipse from a circleDerivation of the formulae  | By substituting functions of this form into differential equations, the equations  are obtained. Multiplying them together gives the formula for  and substituting this formula into either of them gives the formula for  To achieve a prescribed starting point  for the motion, the equations  are solved, using the formula for  to eliminate either  or Students apply this theory to determine the time behaviour of the systems used to motivate the study of these differential equations, or to determine some of their characteristics, such as the frequency of their oscillation. Thus, in relation to the examples given above: the frequency of the radio signal generated by a given capacitor–coil circuit could be determined, given the values of  (inductance) and (capacitance); or the entire time behaviour of the voltage and current could be determined, given their starting values. In a Cartesian coordinate system, with axes chosen for voltage and current, graphics software could be used to plot the elliptical motion of the point that represents the pair. |
| Are the solutions constructed the only ones possible? * There are no other solutions
* Any two motions satisfying a given system of differential equations that start at the same point coincide for all time
 | This idea is a fitting topic for further work in an investigation. Students may be encouraged to see the strength of this idea by noticing that moving points of the form satisfy  This would mean, however, that they must have the form which provides another proof of the addition of angles formulae. |
| Electronic technology is used to explore the motion of points under a given law of interaction by moving the point according to the velocity that the law prescribes for its current position | For two-dimensional systems, computer packages can trace the motion of points under a given law of interaction by moving the point according to the velocity that the law prescribes for its current position. Students experiment with the circular and elliptical motion systems in this subtopic, and extend to more complex systems, noting the new phenomena that can occur (spirals, hyperbolae, limit cycles, etc.).In an investigation students could develop a sense of the phenomena that current mathematical investigations seek to explain, and the kinds of real problems in which they arise by generalising the systems studied in this subtopic to  This would allow for a wide range of behaviour, depending on the choice of coefficients. By graphing slope fields, numerical solution curves, and analytical solutions, students develop an appreciation of the role of initial conditions and the system’s coefficients.Students use electronic technology to plot slope fields and solution curve approximations for the more general systems of the typeSuch systems have general solutions of one of the forms:.Using initial conditions, students determine particular solutions to a system given the form of the relevant general solution. |

Assessment Scope and Requirements

All Stage 2 subjects have a school assessment component and an external assessment component.

Teachers design a set of school assessments that enable students to demonstrate the knowledge, skills, and understanding they have developed to meet the learning requirements of the subject. These assessments provide students’ evidence of learning in the school assessment component.

Evidence of Learning

The following assessment types enable students to demonstrate their learning in Stage 2 Specialist Mathematics:

School Assessment (70%)

* Assessment Type 1: Skills and Applications Tasks (45%)
* Assessment Type 2: Folio (25%)

External Assessment (30%)

* Assessment Type 3: Examination (30%).

Students should provide evidence of their learning through nine to twelve assessments, including the external assessment component. Students undertake:

* at least six skills and applications tasks
* at least two investigations for the folio
* one examination.

Assessment Design Criteria

The assessment design criteria are based on the learning requirements and are used by:

* teachers to clarify for the student what he or she needs to learn
* teachers and assessors to design opportunities for the student to provide evidence of his or her learning at the highest possible level of achievement.

The assessment design criteria consist of specific features that:

* students should demonstrate in their learning
* teachers and assessors look for as evidence that students have met the learning requirements.

For this subject the assessment design criteria are:

* mathematical knowledge and skills and their application
* mathematical modelling and problem-solving
* communication of mathematical information.

The specific features of these criteria are listed below.

The set of assessments, as a whole, must give students opportunities to demonstrate each of the specific features by the completion of study of the subject.

Mathematical Knowledge and Skills and Their Application

The specific features are as follows:

MKSA1 Knowledge of content and understanding of mathematical concepts and relationships.

MKSA2 Use of mathematical algorithms and techniques (implemented electronically where appropriate) to find solutions to routine and complex questions.

MKSA3 Application of knowledge and skills to answer questions set in applied and theoretical contexts, including some attempts at proof.

Mathematical Modelling and Problem-solving

The specific features are as follows:

MMP1 Application of mathematical models.

MMP2 Development of solutions to mathematical problems set in applied and theoretical contexts.

MMP3 Interpretation of the mathematical results in the context of the problem.

MMP4 Understanding of the reasonableness and possible limitations of the interpreted results, and recognition of assumptions made.

MMP5 Development and testing of conjectures, with some attempt at proof.

Communication of Mathematical Information

The specific features are as follows:

CMI1 Communication of mathematical ideas and reasoning to develop logical arguments, including some attempt at proof in applied and/or theoretical contexts.

CMI2 Use of appropriate mathematical notation, representations, and terminology.

School Assessment

Assessment Type 1: Skills and Applications Tasks (45%)

Students undertake at least six skills and applications tasks.

Students find solutions to mathematical questions/problems that may:

* be routine, analytical, and/or interpretative
* be posed in familiar and unfamiliar contexts
* require a discerning use of electronic technology.

In setting skills and applications tasks, teachers may provide students with information in written form or in the form of numerical data, diagrams, tables, or graphs. A task should require the student to demonstrate an understanding of relevant mathematical ideas, facts, and relationships.

Students select appropriate algorithms or techniques and relevant mathematical information to find solutions to routine, analytical, and/or interpretative questions/problems. Some of these problems should be set in a personal, global, or historical context.

Students provide explanations and arguments, and use notation, terminology, and representation correctly throughout the task. They may be required to use electronic technology appropriately to aid and enhance the solution of some questions/problems.

Skills and applications tasks are undertaken under the direct supervision of a teacher.

For this assessment type, students provide evidence of their learning in relation to the following assessment design criteria:

* mathematical knowledge and skills and their application
* mathematical modelling and problem-solving
* communication of mathematical information.

Assessment Type 2: Folio (25%)

A folio consists of at least two investigations.

Note: Teachers may need to provide support and clear directions with the first investigation. However, subsequent investigation(s) should be less directed and set within more open-ended contexts.

Students investigate mathematical relationships, concepts, or problems, which may be set in an applied context. The subject of the investigation may be derived from one or more subtopics, although it can also relate to a whole topic or across topics.

An investigation may be initiated by a student, a group of students, or the teacher. In some instances teachers may give students a clear, detailed, and sequential set of instructions for part of the investigation or to initiate the investigation. In other situations teachers may provide broad guidelines allowing the student or group of students sufficient scope to develop themes or aspects of their own choice. Teachers should be prepared to give some direction about the appropriateness of each student’s choice and to guide and support students’ progress in an investigation.

Students are encouraged to demonstrate their use of problem-solving strategies as well as their knowledge, skills, and understanding in the investigation. The generation of data and the exploration of patterns and structures, or changing parameters, may provide an important focus. From these, students may recognise different patterns or structures. Notation, terminology, forms of representation of information gathered or produced, calculations, and results are important considerations.

Students interpret and justify results, summarise, and draw conclusions. Students are required to give appropriate explanations and arguments in a report. An investigation may require the use of electronic technology.

An investigation provides an opportunity for students to work cooperatively to achieve the learning requirements. When an investigation is undertaken by a group, each student must submit an individual report.

A completed investigation should include:

* an introduction that outlines the problem to be explored, including its significance, its features, and the context
* the method required to find a solution, in terms of the mathematical model or strategy to be used
* the appropriate application of the mathematical model or strategy, including
* the generation or collection of relevant data and/or information, with details of the process of collection
* mathematical calculations and results, and appropriate representations
* the analysis and interpretation of results
* reference to the limitations of the original problem
* a statement of the results and conclusions in the context of the original problem
* appendices and a bibliography, as appropriate.

The format of an investigation may be written, oral, or multimodal.

For this assessment type, students provide evidence of their learning in relation to the following assessment design criteria:

* mathematical knowledge and skills and their application
* mathematical modelling and problem-solving
* communication of mathematical information.

External Assessment

Assessment Type 3: Examination (30%)

Students undertake a 3-hour external examination based on the key questions and key ideas outlined in the five topics and their subtopics. The considerations for developing teaching and learning strategies are provided as a guide only, although applications described under this heading may provide useful contexts for examination questions/problems.

The examination consists of three sections: the first focuses on knowledge and routine skills and applications; the second focuses on more complex questions/problems; and the third focuses on investigative questions/problems. Some questions/problems may require students to interrelate their knowledge, skills, and understanding in more than one topic. The skills and understanding developed through the investigations are also assessed in the examination.

Students must have access to approved electronic technology during the external examination. However, students need to be discerning in their use of electronic technology to find solutions to questions/problems in examinations.

For this assessment type, students provide evidence of their learning in relation to the following assessment design criteria:

* mathematical knowledge and skills and their application
* mathematical modelling and problem-solving
* communication of mathematical information.

Performance Standards

The performance standards describe five levels of achievement, A to E.

Each level of achievement describes the knowledge, skills, and understanding that teachers and assessors refer to in deciding, on the basis of the evidence provided, how well a student has demonstrated his or her learning.

During the teaching and learning program the teacher gives students feedback on, and makes decisions about, the quality of their learning, with reference to the performance standards.

Students can also refer to the performance standards to identify the knowledge, skills, and understanding that they have demonstrated and those specific features that they still need to demonstrate to reach their highest possible level of achievement.

At the student’s completion of study of each school assessment type, the teacher makes a decision about the quality of the student’s learning by:

* referring to the performance standards
* assigning a grade between A and E for the assessment type.

A SACE Board school assessment grade calculator is available on the SACE website (www.sace.sa.edu.au) to combine the grades for the school assessment.

In the external assessment, assessors use the performance standards to make a decision about the quality of students’ learning, based on the evidence provided.

The student’s school assessment and external assessment are combined for a final result, which is reported as a grade between A and E.

Performance Standards for Stage 2 Specialist Mathematics

| - | Mathematical Knowledge and Skills and Their Application | Mathematical Modelling and Problem-solving | Communication of Mathematical Information |
| --- | --- | --- | --- |
| A | Comprehensive knowledge of content and understanding of concepts and relationships.Appropriate selection and use of mathematical algorithms and techniques (implemented electronically where appropriate) to find efficient solutions to complex questions. Highly effective and accurate application of knowledge and skills to answer questions set in applied and theoretical contexts, especially proof. | Development and effective application of mathematical models.Complete, concise, and accurate solutions to mathematical problems set in applied and theoretical contexts.Concise interpretation of the mathematical results in the context of the problem.In-depth understanding of the reasonableness and possible limitations of the interpreted results, and recognition of assumptions made.Development and testing of valid conjectures, with proof. | Highly effective communication of mathematical ideas and reasoning to develop logical arguments, especially proof in applied and theoretical contexts.Proficient and accurate use of appropriate notation, representations, and terminology. |
| B | Some depth of knowledge of content and understanding of concepts and relationships.Use of mathematical algorithms and techniques (implemented electronically where appropriate) to find some correct solutions to complex questions.Accurate application of knowledge and skills to answer questions set in applied and theoretical contexts, including proof. | Attempted development and appropriate application of mathematical models.Mostly accurate and complete solutions to mathematical problems set in applied and theoretical contexts.Complete interpretation of the mathematical results in the context of the problem.Some depth of understanding of the reasonableness and possible limitations of the interpreted results, and recognition of assumptions made.Development and testing of reasonable conjectures, with substantial attempt at proof. | Effective communication of mathematical ideas and reasoning to develop mostly logical arguments, including proof in applied and theoretical contexts.Mostly accurate use of appropriate notation, representations, and terminology. |
| C | Generally competent knowledge of content and understanding of concepts and relationships.Use of mathematical algorithms and techniques (implemented electronically where appropriate) to find mostly correct solutions to routine questions.Generally accurate application of knowledge and skills to answer questions set in applied and theoretical contexts, including some attempts at proof. | Appropriate application of mathematical models.Some accurate and generally complete solutions to mathematical problems set in applied and theoretical contexts.Generally appropriate interpretation of the mathematical results in the context of the problem.Some understanding of the reasonableness and possible limitations of the interpreted results, and some recognition of assumptions made.Development and testing of reasonable conjectures, with some attempt at proof. | Appropriate communication of mathematical ideas and reasoning to develop some logical arguments, including some attempt at proof in applied and/or theoretical contexts.Use of generally appropriate notation, representations, and terminology, with some inaccuracies. |
| D | Basic knowledge of content and some understanding of concepts and relationships.Some use of mathematical algorithms and techniques (implemented electronically where appropriate) to find some correct solutions to routine questions.Sometimes accurate application of knowledge and skills to answer questions set in applied or theoretical contexts. | Application of a mathematical model, with partial effectiveness.Partly accurate and generally incomplete solutions to mathematical problems set in applied or theoretical contexts.Attempted interpretation of the mathematical results in the context of the problem.Some awareness of the reasonableness and possible limitations of the interpreted results.Attempted development or testing of a reasonable conjecture. | Some appropriate communication of mathematical ideas and reasoning.Some attempt to use appropriate notation, representations, and terminology, with occasional accuracy. |
| E | Limited knowledge of content.Attempted use of mathematical algorithms and techniques (implemented electronically where appropriate) to find limited correct solutions to routine questions.Attempted application of knowledge and skills to answer questions set in applied or theoretical contexts, with limited effectiveness. | Attempted application of a basic mathematical model.Limited accuracy in solutions to one or more mathematical problems set in applied or theoretical contexts.Limited attempt at interpretation of the mathematical results in the context of the problem.Limited awareness of the reasonableness and possible limitations of the results.Limited attempt to develop or test a conjecture. | Attempted communication of emerging mathematical ideas and reasoning.Limited attempt to use appropriate notation, representations, or terminology, and with limited accuracy. |

Assessment Integrity

The SACE Assuring Assessment Integrity Policy outlines the principles and processes that teachers and assessors follow to assure the integrity of student assessments. This policy is available on the SACE website (www.sace.sa.edu.au) as part of the SACE Policy Framework.

The SACE Board uses a range of quality assurance processes so that the grades awarded for student achievement, in both the school assessment and the external assessment, are applied consistently and fairly against the performance standards for a subject, and are comparable across all schools.

Information and guidelines on quality assurance in assessment at Stage 2 are available on the SACE website (www.sace.sa.edu.au).

Support Materials

Subject-specific Advice

Online support materials are provided for each subject and updated regularly on the SACE website (www.sace.sa.edu.au). Examples of support materials are sample learning and assessment plans, annotated assessment tasks, annotated student responses, and recommended resource materials.

Advice on Ethical Study and Research

Advice for students and teachers on ethical study and research practices is available in the guidelines on the ethical conduct of research in the SACE on the SACE website (www.sace.sa.edu.au).